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On Jackson's Theorem

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If f is a continuous function on $[a, b]$ and $\omega(f, h)$ is its modulus of continuity, then the classical theorem of Jackson ([*1*], p. 15) states that for each positive n there is an algebraic polynomial P of degree $\leqslant n$ such that

$$\max_{a \leqslant x \leqslant b} |f(x) - P(x)| \leqslant C\omega(f, 1/n) \tag{1}$$

where C is a constant independent of f and n. Jackson's proof of this theorem consisted of proving the analogous result for approximation of 2π-periodic functions by trigonometric polynomials and using a standard transformation to obtain (1). At the Oberwolfach Conference on Approximation Theory in 1963 ([*2*], p. 180), P. L. Butzer has pointed out the desirability for a direct proof of (1). Subsequently, G. Freud [*3*] and R. B. Saxena [*4*] have both constructed interpolation procedures which lead to (1). Freud's results were later generalized by M. Sallay [*7*].

It is natural to ask whether one can obtain Jackson's theorem by considering convolution with non-negative algebraic polynomials. For example, the Landau polynomials

$$C_n \int_{-1}^{1} f(t)(1 - (t - x)^2)^n \, dt, \qquad 1/C_n = \int_{-1}^{1} (1 - x^2)^n \, dx$$

converge uniformly to f on $[-\delta, \delta]$ if $\delta < 1$, provided that f is continuous on $[-1, 1]$. However, they do not provide the order of approximation (1).

In this paper, we shall consider the nonnegative polynomials

$$\Lambda_n(t) = C_n \frac{(P_{2n}(t))^2}{(t^2 - x_{n+1}^2)^2}, \qquad 1/C_n = \int_{-1}^{1} \frac{(P_{2n}(t))^2 \, dt}{(t^2 - x_{n+1}^2)^2},$$

where $P_{2n}(t)$ is the Legendre polynomial of degree $2n$, and x_{n+1} is its smallest positive zero. Our main result is the following

THEOREM 1. *If f is a continuous function on $[-\frac{1}{2}, \frac{1}{2}]$ and $f(-\frac{1}{2}) = f(\frac{1}{2}) = 0$, then the polynomial*

$$L_n(f, x) = \int_{-1/2}^{1/2} f(t) \, \Lambda_n(t - x) \, dt$$

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of degree $\leqslant 4n-4$, *satisfies*

$$\max_{-\frac{1}{2}\leqslant x\leqslant\frac{1}{2}} |f(x)-L_n(f,x)| \leqslant 40\omega(f,1/n), \qquad n=1,2,3,\ldots. \tag{2}$$

It is readily seen that the restriction that f vanishes at the end-points is necessary to guarantee uniform convergence of polynomials which arise from convoluting f with an even polynomial. Essentially, this is due to the fact that the integrals of the kernels over $[-1,0]$ and $[0,1]$ are only $\frac{1}{2}$. It is easy to obtain Jackson's theorem from Theorem 1. The polynomial

$$\bar{L}_n(f,x)=l(x)+L_n(f-l,x),$$

where

$$l(x)=f(-\tfrac{1}{2})+[f(\tfrac{1}{2})-f(-\tfrac{1}{2})](x+\tfrac{1}{2}),$$

provides the estimate (1) for an arbitrary continuous function when $a=-\frac{1}{2}$, $b=\frac{1}{2}$.

We shall need the following.

LEMMA 1. *Let* $\gamma(t)$ *have total variation* $\leqslant A$ *on* $[-\frac{1}{2},\frac{1}{2}]$, $A\geqslant 1$.

If f *is continuous on* $[-\frac{1}{2},\frac{1}{2}]$ *and* $f(-\frac{1}{2})=f(\frac{1}{2})=0$, *then*

$$\left|\int_{-1/2}^{1/2} f(t)\,d\gamma(t)\right| \leqslant 4A\omega(f,\delta),$$

where

$$\delta=\int_{-1/2}^{1/2}|\gamma(t)|\,dt.$$

Proof. Let us first consider an arbitrary function g in $\mathrm{Lip}_M 1$ which vanishes at $-\frac{1}{2}$ and $\frac{1}{2}$. We have

$$g(x)=\int_{-1/2}^{x} g'(t)\,dt, \qquad \text{with } |g'(t)|\leqslant M \text{ a.e. on } [-\tfrac{1}{2},\tfrac{1}{2}].$$

Thus,

$$\left|\int_{-1/2}^{1/2} g(t)\,d\gamma(t)\right| = \left|\int_{-1/2}^{1/2} g'(t)\gamma(t)\,dt\right| \leqslant M\int_{-1/2}^{1/2}|\gamma(t)|\,dt = M\delta.$$

Suppose now that f is an arbitrary continuous function on $[-\frac{1}{2},\frac{1}{2}]$. We can find a concave modulus of continuity $\omega_1(t)$ (see [5], p. 45) satisfying

$$\omega(f,h)\leqslant\omega_1(h)\leqslant 2\omega(f,h), \qquad 0\leqslant h\leqslant 1. \tag{4}$$

We shall use the following result on approximation by functions in $\mathrm{Lip}_M 1$ (see [5], pp. 122–123).

PROPOSITION 1. *Let* $0<\eta\leqslant 1$ *and let* ω_1 *be a concave modulus of continuity. Then, there exists an* $M>0$ *such that for each continuous function* f *whose modulus of continuity* ω *satisfies*

$$\omega(f,h)\leqslant\omega_1(h), \qquad h>0$$

we can find a function g in $\mathrm{Lip}_M 1$ *for which*

$$\max_{-\frac{1}{2}\leqslant x\leqslant \frac{1}{2}} |f(x)-g(x)| \leqslant \omega_1(\eta) - M\eta.$$

Now the rest of the proof of Lemma 1 is simple. Let $\eta = \delta$ in the above proposition. Suppose f satisfies the hypothesis of Lemma 1, and g is the function given by Proposition 1. The function

$$\bar{g}(x) = g(x) - [g(-\tfrac{1}{2}) + (g(\tfrac{1}{2}) - g(-\tfrac{1}{2}))(x+\tfrac{1}{2})]$$

is in $\mathrm{Lip}_{2M} 1$ and satisfies

$$\begin{aligned}\max_x |f(x)-\bar{g}(x)| &\leqslant \max_x |f(x)-g(x)| + \max_x |g(x)-\bar{g}(x)| \\ &\leqslant \omega_1(\delta) - M\delta + \omega_1(\delta) - M\delta = 2(\omega_1(\delta) - M\delta), \qquad (5)\end{aligned}$$

since

$$|g(-\tfrac{1}{2})| \leqslant \omega_1(\delta) - M\delta \quad \text{and} \quad |g(\tfrac{1}{2})| \leqslant \omega_1(\delta) - M\delta.$$

Thus,

$$\begin{aligned}\left|\int_{-1/2}^{1/2} f(t)\,d\gamma(t)\right| &\leqslant \int_{-1/2}^{1/2} |f(t)-\bar{g}(t)|\cdot|d\gamma(t)| + \left|\int_{-1/2}^{1/2} \bar{g}(t)\,d\gamma(t)\right| \\ &\leqslant (2\omega_1(\delta) - 2M\delta)A + 2M\delta \leqslant 2A\omega_1(\delta),\end{aligned}$$

where the first term was estimated by (5) and the second term by (3). The proof is complete by invoking (4).

We note two elementary properties of the polynomials P_{2n} which can be found in [6].

PROPOSITION 2. ([6], *p.* 121). *If* x_{n+1} *is the smallest positive zero of* P_{2n}, *then* $x_{n+1} \leqslant 2/n$.

PROPOSITION 3. (*The Gauss Quadrature Formula* [6], *p.* 97.) *Let* $x_1, \ldots, x_{2n}$ *be the zeros of* P_{2n} *written in increasing order. Then, there exist real positive constants* $A_k^{(n)}$, $k = 1, 2, \ldots, 2n$, *such that for each polynomial* Q *of degree* $\leqslant 4n-1$, *we have*

$$\int_{-1}^{1} Q(t)\,dt = \sum_{k=1}^{2n} A_k^{(n)} Q(x_k).$$

Note: *Since* P_{2n} *is an even polynomial,* $x_n = -x_{n+1}$ *and* $A_n^{(n)} = A_{n+1}^{(n)}$.

Now to the proof of Theorem 1. Let u denote the Dirac measure having unit mass at 0. If f is continuous on $[-\frac{1}{2}, \frac{1}{2}]$, we have the representation

$$f(x) - L_n(f,x) = \int_{-1/2}^{1/2} f(t)[du(t-x) - \Lambda_n(t-x)\,dt] = \int_{-1/2}^{1/2} f(t)\,d\gamma_n(t-x),$$

where

$$\gamma_n(t) = u(t) - \int_{-1}^{t} \Lambda_n(x)\,dx.$$

Also, for $|x| \leqslant \frac{1}{2}$,

$$\int_{-1/2}^{1/2} |\gamma_n(t-x)|\,dt \leqslant \int_{-1}^{1} |\gamma_n(t)|\,dt. \tag{6}$$

If we integrate by parts, we find

$$\int_{-1}^{1} |\gamma_n(t)|\,dt = \int_{-1}^{1} |t|\,\Lambda_n(t)\,dt.$$

Now,

$$\int_{-1/n}^{1/n} |t|\,\Lambda_n(t)\,dt \leqslant 1/n \int_{-1/n}^{1/n} \Lambda_n(t)\,dt \leqslant 1/n.$$

Using Proposition 3 and observing that Λ_n is a polynomial of degree $4n-4$, we have

$$\begin{aligned}
\int_{[-1,1]-[-n^{-1},n^{-1}]} |t|\,\Lambda_n(t)\,dt \leqslant n \int_{-1}^{1} t^2 \Lambda_n(t)\,dt
&= n \sum_{k=1}^{2n} A_k^{(n)} x_k{}^2 \Lambda_n(x_n) \\
&= n(A_n^{(n)} x_n{}^2 \Lambda_n(x_n) + A_n^{(n+1)} x_{n+1}^2) \\
&= n x_{n+1}^2 (A_n^{(n+1)} \Lambda_n(x_{n+1}) + A_n^{(n)} \Lambda_n(x_n)) \\
&= n x_{n+1}^2 \int_{-1}^{1} \Lambda_n(t)\,dt = n x_{n+1}^2.
\end{aligned}$$

Thus from Proposition 2, we find

$$\int_{[-1,1]-[-n^{-1},n^{-1}]} |t|\,\Lambda_n(t)\,dt \leqslant n x_{n+1}^2 \leqq 4/n.$$

By virtue of (6), we have for $|x| \leqslant \frac{1}{2}$,

$$\int_{-1/2}^{1/2} |\gamma_n(t-x)|\,dt \leqslant \int_{-1}^{1} |t|\,\Lambda_n(t)\,dt \leqslant 1/n + 4/n = 5/n. \tag{7}$$

Finally, since the total variation of $\gamma_n(t-x)$ is $\leqslant 2$ on $[-\frac{1}{2}, \frac{1}{2}]$, and $\omega(f, 5/n) \leqslant 5\omega(f, 1/n)$, Theorem 1 follows immediately from Lemma 1 and from (7).

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